Predicate and Relation Lifting for Parametric Algebraic Specifications

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This talk is about

Bisimulation and Invariants for Coalgebras

of a specific class of functors.

Elem : **Loose Specification**

Operation

equal : $\mathrm{Elem} \times \mathrm{Elem} \longrightarrow \mathrm{bool}$

End Elem

FSetDel[T : Elem] : **Initial Specification**

Operations

empty : $\mathrm{FSetDel}$

add : $\mathrm{FSetDel} \times T \longrightarrow \mathrm{FSetDel}$

del : $\mathrm{FSetDel} \times T \longrightarrow \mathrm{FSetDel}$

Variables

$x : \mathrm{FSetDel}; \quad t, t_1, t_2 : T$

Axioms

add_comm : $\mathrm{add}(\mathrm{add}(x, t_1), t_2) = \mathrm{add}(\mathrm{add}(x, t_2), t_1)$

add_idem : $\mathrm{add}(\mathrm{add}(x, t), t) = \mathrm{add}(x, t)$

del_empty : $\mathrm{del}(\mathrm{empty}, t) = \mathrm{empty}$

del_add_eq : $\mathrm{equal}(t_1, t_2) = \mathrm{true} \Longrightarrow \mathrm{del}(\mathrm{add}(x, t_1), t_2) = \mathrm{del}(x, t_2)$

del_add_neq : $\mathrm{equal}(t_1, t_2) = \mathrm{false} \Longrightarrow \mathrm{del}(\mathrm{add}(x, t_1), t_2) = \mathrm{add}(\mathrm{del}(x, t_2), t_1)$

End FSetDel

Example II

Type : **Loose Specification**

End Type

FSet[T : Type] : **Initial Specification**

Operations

empty : FSet

add : $\mathrm{FSet} \times T \longrightarrow \mathrm{FSet}$

Variables

$x : \mathrm{FSet}; \quad t, t_1, t_2 : T$

Axioms

add_comm : $\mathrm{add}(\mathrm{add}(x, t_1), t_2) = \mathrm{add}(\mathrm{add}(x, t_2), t_1)$

add_idem : $\mathrm{add}(\mathrm{add}(x, t), t) = \mathrm{add}(x, t)$

End FSet

- The specifications FSet/FSetDel define functors $[\![\text{FSet}]\!]$/$[\![\text{FSetDel}]\!]$ that map models of Type/Elem to models of FSet/FSetDel (made precise later)

Consider Coalgebras $X \longrightarrow [\![\text{FSetDel}]\!](X)$

To define such a coalgebra we have to:

- select a carrier X
- define an equality relation on X to make it a model of Elem
- form $[\![\text{FSetDel}]\!](X)$
- define coalgebra mapping $X \longrightarrow [\![\text{FSetDel}]\!](X)$

What are $[\![\text{FSetDel}]\!]$–Bisimulations?

What are $[\![\text{FSetDel}]\!]$–Invariants?

- definition of the paper works for FSet
- however, it fails for FSetDel
- Proposition 3.4. in the paper is wrong
  (please don't cite it)
- improved definitions in this talk
- get a corrected version from
  http://wwwtcs.inf.tu-dresden.de/~tews/science.html

Contents

Bisimulations according to Aczel/Mendler [1]

$$\begin{tikzcd}
X \arrow[d, "c"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d] & Y \arrow[d, "d"] \\
[\![\mathsf{FSetDel}]\!](X) & [\![\mathsf{FSetDel}]\!](R) \arrow[l, "{[\![\mathsf{FSetDel}]\!](\pi_1)}"'] \arrow[r, "{[\![\mathsf{FSetDel}]\!](\pi_2)}"] & [\![\mathsf{FSetDel}]\!](Y)
\end{tikzcd}$$

- usually impossible to equip R with an equality relation such that $\pi_1 : R \longrightarrow X$ and $\pi_2 : R \longrightarrow Y$ are Elem-Morphisms:

$$\begin{array}{ccccc}
x = x & \Longrightarrow & (x, y_1) = (x, y_2) & \Longrightarrow & y_1 = y_2 \\
x_1 \neq x_2 & \Longrightarrow & (x_1, y) \neq (x_2, y) & \Longrightarrow & y \neq y
\end{array}$$

Bisimulations according to Hermida/Jacobs [6]

- need to define relation lifting for $[\![\mathsf{FSetDel}]\!]$
- equation $\mathrm{Rel}(F)(R) = \coprod_{\langle F(\pi_1), F(\pi_2) \rangle} F(R)$ cannot be used

Bisimulations via Cospans

$R \subseteq X \times Y$ is a bisimulation if there exists z, f, g:

$$
\begin{tikzcd}
X \arrow[d, "c"'] \arrow[drr, "f"] & & & & Y \arrow[dll, "g"'] \arrow[d, "d"] \\
\llbracket \mathsf{FSetDel} \rrbracket (X) \arrow[drr, "\llbracket \mathsf{FSetDel} \rrbracket (f)"] & & Z \arrow[d, "z"'] & & \llbracket \mathsf{FSetDel} \rrbracket (Y) \arrow[dll, "\llbracket \mathsf{FSetDel} \rrbracket (g)"'] \\
& & \llbracket \mathsf{FSetDel} \rrbracket (Z) & &
\end{tikzcd}
$$

such that $x \, R \, y \quad \Longleftrightarrow \quad f \, x = g \, y$

- resulting notion of bisimulation does not agree with intuition (see Example 2.4 in the paper)

In general

- ignore syntactic issues
- work with single sorted specifications

Consider a parameter specification $\mathcal{T}$

- neither algebraic nor coalgebraic
- loose semantics
- given as category $\mathrm{Mod}(\mathcal{T})$ of models of $\mathcal{T}$

Example – Elem

- models of Elem are pairs $\langle E, eq \rangle$ with $eq : E \times E \longrightarrow bool$
- morphisms are functions that preserve and *reflect* equality: $eq_2(f\ e_1, f\ e_2) = eq_1(e_1, e_2)$

Iterated polynomial functors [5] $\mathbf{Set}^n \longrightarrow \mathbf{Set}$

$$F(X_1,\ldots,X_n) \;=\; A \;\mid\; X_i \;\mid\; F_1(\cdots) \overset{+}{\times} F_2(\cdots) \;\mid\; A \Rightarrow F_1(\cdots) \;\mid\; \mu/\nu\, X \,.\, F_1(\ldots,X,\ldots)$$

Consider an algebraic specification $\mathcal{S}$

- parametric in $\mathcal{T}$, single sorted, initial semantics
- Signature given as $\mathit{Sig} : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$ (an iterated polynomial functor)
- signature models are algebras: $\mathit{Sig}(U,X) \longrightarrow X$
- models are pairs $\langle T, a\rangle$ with $a : \mathit{Sig}(|T|,X) \longrightarrow X$
- morphisms between models $\begin{array}{c} T_1 \\ \downarrow f \\ T_2 \end{array}$ $\qquad$ $\begin{array}{ccc} \mathit{Sig}(|T_1|,X) & \xrightarrow{a} & X \\ {\scriptstyle \mathit{Sig}(f,g)}\downarrow & & \downarrow{\scriptstyle g} \\ \mathit{Sig}(|T_2|,Y) & \xrightarrow{b} & Y \end{array}$
- forgetful functor $\mathcal{U} : \mathrm{Mod}(\mathcal{S}) \longrightarrow \mathrm{Mod}(\mathcal{T})$

- semantics $[\![\mathcal{S}]\!] \dashv \mathcal{U}$ with $\mathcal{U} \circ [\![\mathcal{S}]\!] = \mathsf{Id}$
- $[\![\mathcal{S}]\!](T)$ is constructed as quotient $\mu X \,.\, \mathit{Sig}(|T|,X) \;/\; Q_T$,
  where Q_T is the least congruence relation implied by the axioms of $\mathcal{S}$
- works for equational Horn formulas [9]

Semantics of the specification FSetDel

- Functor $F(T, X) = \mathbf{1} + (X \times T) + (X \times T)$
- signature model $\gamma : F(T, X) \longrightarrow X$ gives
  $\mathsf{empty}_\gamma = \gamma \circ \kappa_1$, $\mathsf{add}_\gamma = \gamma \circ \kappa_2$, ...
- Initial $F(T, -)$ algebra $\mu X \,.\, F(T, X)$:
  - Carrier are lists over $T \times \mathit{bool}$
    (t, true) encodes an $\mathsf{add}(-, t)$, (t, false) encodes a $\mathsf{del}(-, t)$
  - predicate and relation lifting [5] for $P \subseteq T$ and $R \subseteq T_1 \times T_2$:

$$\mathrm{Pred}(\mu X \,.\, F(-, X))(P)(l) \quad = \quad P \text{ holds on every element of } T \text{ in } l$$

$$\mathrm{Rel}(\mu X \,.\, F(-, X))(R)(l_1, l_2) \quad = \quad \begin{cases} l_1 \text{ and } l_2 \text{ have the same length} \\ R \text{ holds on corresponding pairs} \end{cases}$$

- $[\![\mathsf{FSetDel}]\!](T)$ are sets over $|T|$ (with respect to the provided equality on T)

Definition 1 (to be improved later)
Let $\mathcal{T}$ and $\mathcal{S}$ as before.

$$\begin{aligned} \mathrm{Pred}(\llbracket\mathcal{S}\rrbracket)(P) &= \coprod_{q_{Q_T}} \mathrm{Pred}(\mu X \,.\, \mathit{Sig}(-, X))(P) \\ \mathrm{Rel}(\llbracket\mathcal{S}\rrbracket)(R) &= \coprod_{q_{Q_T} \times q_{Q_V}} \mathrm{Rel}(\mu X \,.\, \mathit{Sig}(-, X))(R) \end{aligned}$$

Here $\coprod_{q_{Q_T}}$ is the image along the quotient map q_{Q_T} used for factoring the initial Sig–algebra

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- Let T_1, T_2 be models of T and $s \subseteq |T_1|$ and $r \subseteq |T_2|$
- $\mathrm{Pred}(\llbracket\mathsf{FSet}\rrbracket)(P)(s) \iff \begin{cases} \text{there exists a list representation } l_s \text{ of } s \\ \text{such that } P \text{ holds on all elements in } l_s \end{cases}$
- $\mathrm{Rel}(\llbracket\mathsf{FSet}\rrbracket)(R)(s, r) \iff \begin{cases} \text{there exist lists } l_s \text{ and } l_r \text{ of equal length} \\ \text{such that } R \text{ holds on corrsponding elements} \end{cases}$
- For FSet these liftings coincide the liftings of the finite powerset functor [7]
- For FSetDel they don't!

- Operations like **del** can spoil the liftings
- Consider a relation R with $a\,R\,b$ and $a\,R\,c$
- in FSetDel:
  $\mathsf{del}(\mathsf{add}(\mathsf{empty}, a), a) = \mathsf{empty}$
  $\mathsf{del}(\mathsf{add}(\mathsf{empty}, c), b) = \mathsf{add}(\mathsf{empty}, \mathsf{c})$
  therefore $(\mathsf{empty}, \mathsf{add}(\mathsf{empty}, c)) \in \mathrm{Rel}([\![\mathsf{FSetDel}]\!])(R)$
- However $(\emptyset, \{c\}) \notin \mathrm{Rel}(\mathcal{P})(R)$

Property	polynomial functors	iterated pol. functors	$[\![\mathcal{S}]\!]$
monotonicity	$\surd$	$\surd$	$\surd$
truth/equality	$\surd$	$\surd$	$\surd$
fibred	$\surd$	$\surd$	$\lightning$
cofibred	$\surd$	?	$\surd^*$
opposite	$\surd$	?	$\surd^*$
Unions	$\subseteq$	?	$\subseteq^\star$
intersection	$\surd$	?	$\lightning$
composition	$\surd$	?	$\lightning$

$-^*$ holds if the property holds for the liftings of $\mu X \,.\, \mathsf{Sig}(-, X)$

Proposition 2 (Monotonicity) *Let $P_1 \subseteq P_2$ and $R_1 \subseteq R_2$:*

$$\begin{array}{rcl} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(P_1) & \subseteq & \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(P_2) \\ \mathrm{Rel}(\llbracket \mathcal{S} \rrbracket)(R_1) & \subseteq & \mathrm{Rel}(\llbracket \mathcal{S} \rrbracket)(R_2) \end{array}$$

Proof
$$\begin{array}{rcl} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(P_1) & = & \coprod_q \mathrm{Pred}(\mu X \,.\, \mathit{Sig}(-, X))(P_1) \\ & \subseteq & \coprod_q \mathrm{Pred}(\mu X \,.\, \mathit{Sig}(-, X))(P_2) \end{array}$$
□

Proposition 3 (Truth and Equality)

$$\begin{array}{rcl} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(\top_{|T|}) & = & \top_{\llbracket \mathcal{S} \rrbracket(T)} \\ \mathrm{Rel}(\llbracket \mathcal{S} \rrbracket)(\mathrm{Eq}(|T|)) & = & \mathrm{Eq}(\llbracket \mathcal{S} \rrbracket(T)) \end{array}$$

Proof
$$\begin{array}{rcl} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(\top_X) & = & \coprod_{q_Q} \mathrm{Pred}(\mu X \,.\, \mathit{Sig}(-, X))(\top_X) \\ & = & \coprod_{q_Q} \top_{\mu X \,.\, \mathit{Sig}(X,X)} \\ & = & \top_{\mu X \,.\, \mathit{Sig}(X,X)/Q} \\ & = & \top_{\llbracket \mathcal{S} \rrbracket(X)} \end{array}$$
□

Proposition 4 (Opposite relation)
Let T, V be models of $\mathcal{T}$ and let $R \subseteq |T| \times |V|$. Assume:

$$\big(\, \mathrm{Rel}(\mu X\,.\,\mathit{Sig}(-,X))(R)\,\big)^{\mathrm{op}} \quad = \quad \mathrm{Rel}(\mu X\,.\,\mathit{Sig}(-,X))(R^{\mathrm{op}})$$

Then we have also: $$\big(\, \mathrm{Rel}([\![\mathcal{S}]\!])(R)\,\big)^{\mathrm{op}} \quad = \quad \mathrm{Rel}([\![\mathcal{S}]\!])(R^{\mathrm{op}})$$

Proof Note that $(\coprod_{f\times g}(R))^{\mathrm{op}} = \coprod_{g\times f} R^{\mathrm{op}}$:

$$\begin{aligned}
\big(\, \mathrm{Rel}([\![\mathcal{S}]\!])(R)\,\big)^{\mathrm{op}} &= \big(\, \textstyle\coprod_{q_T\times q_V} \mathrm{Rel}(\mu X\,.\,\mathit{Sig}(-,X))(R)\,\big)^{\mathrm{op}} \\
&= \textstyle\coprod_{q_V\times q_T} \big(\, \mathrm{Rel}(\mu X\,.\,\mathit{Sig}(-,X))(R)\,\big)^{\mathrm{op}} \\
&= \textstyle\coprod_{q_V\times q_T} \mathrm{Rel}(\mu X\,.\,\mathit{Sig}(-,X))(R^{\mathrm{op}}) \\
&= \mathrm{Rel}([\![\mathcal{S}]\!])(R^{\mathrm{op}})
\end{aligned}$$
$\square$

Proposition 5 (Unions)
Let T, V be models of $\mathcal{T}$ and let $(P_i \subseteq |T|)_{i \in I}$ and $(R_i \subseteq |T| \times |V|)_{i \in I}$. Assume:

$$\begin{array}{rcl} \bigcup_i \mathrm{Pred}(\mu X . \mathit{Sig}(-,X))(P_i) & \subseteq & \mathrm{Pred}(\mu X . \mathit{Sig}(-,X))(\bigcup_i P_i) \\ \bigcup_i \mathrm{Rel}(\mu X . \mathit{Sig}(-,X))(R_i) & \subseteq & \mathrm{Rel}(\mu X . \mathit{Sig}(-,X))(\bigcup_i R_i) \end{array}$$

Then it holds

$$\begin{array}{rcl} \bigcup_i \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(P_i) & \subseteq & \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(\bigcup_i P_i) \\ \bigcup_i \mathrm{Rel}(\llbracket \mathcal{S} \rrbracket)(R_i) & \subseteq & \mathrm{Rel}(\llbracket \mathcal{S} \rrbracket)(\bigcup_i R_i) \end{array}$$

Proof Note that $\coprod_f (\bigcup_i P_i) = \bigcup_i \coprod_f P_i$:

$$\begin{array}{rcl} \bigcup_i \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(P_i) & = & \bigcup_i \coprod_q \mathrm{Pred}(\mu X . \mathit{Sig}(-,X))(P_i) \\ & = & \coprod_q \bigcup_i \mathrm{Pred}(\mu X . \mathit{Sig}(-,X))(P_i) \\ & \subseteq & \coprod_q \mathrm{Pred}(\mu X . \mathit{Sig}(-,X))(\bigcup_i P_i) \\ & = & \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(\bigcup_i P_i) \end{array}$$ □

Proposition 6 *Let T_1, T_2, V_1, V_2 be models of $\mathcal{T}$ and let $f : T_1 \longrightarrow T_2$, $g : V_1 \longrightarrow V_2$, $P \subseteq |T_2|$, and $R \subseteq |T_2| \times |V_2|$. Assume*

$$\begin{array}{rcl} \mathrm{Pred}(\mu X . \mathit{Sig}(-, X))(\coprod_f P) & = & \coprod_{\mu X . \mathit{Sig}(f,X)} \mathrm{Pred}(\mu X . \mathit{Sig}(-, X))(P) \\ \mathrm{Rel}(\mu X . \mathit{Sig}(-, X))(\coprod_{f \times g} R) & = & \coprod_{\mu X . \mathit{Sig}(f,X) \times \mu X . \mathit{Sig}(g,X)} \mathrm{Rel}(\mu X . \mathit{Sig}(-, X))(R) \end{array}$$

Then

$$\begin{array}{rcl} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(\coprod_f P) & = & \coprod_{\llbracket \mathcal{S} \rrbracket(f)} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(P) \\ \mathrm{Rel}(\llbracket \mathcal{S} \rrbracket)(\coprod_{f \times g} R) & = & \coprod_{\llbracket \mathcal{S} \rrbracket(f) \times \llbracket \mathcal{S} \rrbracket(g)} \mathrm{Rel}(\llbracket \mathcal{S} \rrbracket)(R) \end{array}$$

Proof Note that $\mathcal{Q}(f) \circ q_R = q_R \circ f$ because $\mathcal{Q} \dashv \mathrm{Eq}$:

$$\begin{array}{rcl} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(\coprod_f P) & = & \coprod_{q_V} \mathrm{Pred}(\mu X . \mathit{Sig}(-, X))(\coprod_f P) \\ & = & \coprod_{q_V} \coprod_{\mu X . \mathit{Sig}(f,X)} \mathrm{Pred}(\mu X . \mathit{Sig}(-, X))(P) \\ & = & \coprod_{\mathcal{Q}(\mu X . \mathit{Sig}(f,X))} \coprod_{q_T} \mathrm{Pred}(\mu X . \mathit{Sig}(-, X))(P) \\ & = & \coprod_{\llbracket \mathcal{S} \rrbracket(f)} \mathrm{Pred}(\llbracket \mathcal{S} \rrbracket)(P) \end{array}$$ $\square$

(For polynomial functors one has $\mathrm{Pred}(F)(P_1 \cap P_2) \; = \; \mathrm{Pred}(F)(P_1) \cap \mathrm{Pred}(F)(P_2)$)

Counterexample

T : **Loose Specification**
  Operation
    k, l : T
End T

S[T : T] : **Initial Specification**
  Operations
    nil : S
    cons : S $\times$ $T \longrightarrow$ S
  Axioms
    a : $\mathrm{cons}(\mathsf{nil}, k) = \mathrm{cons}(\mathsf{nil}, \mathsf{l})$
End S

- Consider $P = \{\mathsf{k}\}$ and $Q = \{\mathsf{l}\}$ such that $P \cap Q = \emptyset$
- $[\mathsf{cons}(\mathsf{nil}, \mathsf{k})] \in \mathrm{Pred}([\![\mathcal{S}]\!])(P)$ and
  $[\mathsf{cons}(\mathsf{nil}, \mathsf{k})] = [\mathsf{cons}(\mathsf{nil}, \mathsf{l})] \in \mathrm{Pred}([\![\mathcal{S}]\!])(Q)$
- $[\mathsf{cons}(\mathsf{nil}, \mathsf{k})] \notin \mathrm{Pred}([\![\mathcal{S}]\!])(P \cap Q)$

(For polynomial functors one has $\mathrm{Rel}(F)(R \cap S) \; = \; \mathrm{Rel}(F)(R) \cap \mathrm{Rel}(F)(S)$)

Counterexample

T : **Loose Specification**
Operation
$\mathsf{k} \quad : \mathsf{T}$
End T

$\mathsf{S}[T : \mathsf{T}]$: **Initial Specification**
Operations
$\mathsf{nil} \quad : \mathsf{S}$
$\mathsf{cons} \quad : \mathsf{S} \times T \longrightarrow \mathsf{S}$
Axioms
$\mathsf{a} \quad : \mathrm{cons}(\mathsf{nil}, k) = \mathsf{nil}$
End S

- Consider R with $(a, \mathsf{k}) \in R$ and $a \neq \mathsf{k}$
- Let $S = \emptyset$
- $([\mathsf{cons}(\mathsf{nil}, a)], [\mathsf{nil}]) \in \mathrm{Rel}([\![\mathcal{S}]\!])(R)$ because $\mathsf{nil} = \mathsf{cons}(\mathsf{nil}, k)$
- $([\mathsf{nil}], [\mathsf{nil}]) \in \mathrm{Rel}([\![\mathcal{S}]\!])(S)$
- However, $R \circ S = \emptyset$ so $([\mathsf{cons}(\mathsf{nil}, a)], [\mathsf{nil}]) \notin \mathrm{Rel}([\![\mathcal{S}]\!])(R \circ S)$

(For polynomial functors one has $\mathrm{Pred}(F)(f^* P) = F(f)^* \mathrm{Pred}(F)(P)$)

Counterexample

T : **Loose Specification**
Operation
$\mathsf{k} : \mathsf{T}$
End T

$\mathsf{S}[T : \mathsf{T}]$: **Initial Specification**
Operations
$\mathsf{nil} : \mathsf{S}$
$\mathsf{cons} : \mathsf{S} \times T \longrightarrow \mathsf{S}$
$\mathsf{m} : \mathsf{S} \times T \longrightarrow \mathsf{S}$
Variables $l : \mathsf{S};\ t : T$
Axioms
$\mathsf{a} : \mathrm{m}(l, t) = \mathrm{cons}(l, \mathsf{k})$
End S

- Consider T with $|T| = \{t_1, t_2\}$ and set $\mathsf{k}_T = t_1$
- Then $f \stackrel{\mathsf{def}}{=} \lambda x : T \,.\, t_1$ is a morphism $\mathsf{T} \longrightarrow \mathsf{T}$
- Let $P = \{t_2\}$ so $f^*(P) = \emptyset$
- $[\mathsf{cons}(\mathsf{nil}, t_1)] \in \mathsf{Pred}([\![\mathcal{S}]\!])(P)$ because $\mathsf{cons}(\mathsf{nil}, t_1) = \mathsf{m}(\mathsf{nil}, t_2)$
- therefore $[\mathsf{cons}(\mathsf{nil}, t_1)] \in ([\![\mathcal{S}]\!](f))^* \,\mathsf{Pred}([\![\mathcal{S}]\!])(P)$
- However $\mathsf{Pred}([\![\mathcal{S}]\!])(f^* P) = \{ [\mathsf{nil}] \}$

- axioms that erase (or introduce) parameter elements pose problems
- Consider canonical forms $C_T \subseteq \mu X \, . \, \mathit{Sig}(|T|, X)$
  such that $\forall x \in \mu X \, . \, \mathit{Sig}(|T|, X) \; . \; [x] \cap C_T \neq \emptyset$
- For FSetDel a term is in canonical form it it is del–free.

Definition 7
Let $\mathcal{T}$ and $\mathcal{S}$ as before and let $P \subseteq |T|$ and $R \subseteq |T| \times |V|$ for models T and V of $\mathcal{T}$.

$$\begin{aligned} \mathrm{Pred}([\![\mathcal{S}]\!])(P) &= \coprod_{q_{Q_T}} \big(\mathrm{Pred}(\mu X \, . \, \mathit{Sig}(-, X))(P) \quad \cap \; C_T\big) \\ \mathrm{Rel}([\![\mathcal{S}]\!])(R) &= \coprod_{q_{Q_T} \times q_{Q_V}} \big(\mathrm{Rel}(\mu X \, . \, \mathit{Sig}(-, X))(R) \quad \cap \; C_T \times C_V\big) \end{aligned}$$

$\coprod_{q_{Q_T}}$ is the quotient map as before.

- For predicate lifting the canonical forms are not necessary.
- Properties of these liftings depend on the subset of the canonical forms.

Property	pol. func.	it. pol. func.	$[\![\mathcal{S}]\!]$	$[\![\mathcal{S}]\!]$ improved
monotonicity	$\surd$	$\surd$	$\surd$	$\surd$
truth/equality	$\surd$	$\surd$	$\surd$	$\surd$
fibred	$\surd$	$\surd$	$\lightning$	$\lightning$
cofibred	$\surd$	?	$\surd^*$	?
opposite	$\surd$	?	$\surd^*$	$\surd^*$
Unions	$\subseteq$	?	$\subseteq^\star$	$\subseteq^\star$
intersection	$\surd$	?	$\lightning$	$\lightning$
composition	$\surd$	?	$\lightning$	$\lightning$

$-^*$ holds if the property holds for the liftings of $\mu X \,.\, \mathsf{Sig}(-, X)$

Conclusion

- Predicate and relation lifting for parametric algebraic specifications
  (that rely on operations of the parameter)
- these liftings define bisimulations and invariants for the derived functors
- Obvious approach
  - factor the liftings of Hensel and Jacobs [5]
    with the (congruence relation derived from the) axioms
  - fails on many specifications
- need to consider subsets of *canonical* terms (best normal forms)
- liftings and its properties depend on the choice of the canonical terms
- new liftings have some nice properties (monotone, unions)
  however other properties fail (fibredness, intersection, composition)

Related Work

- CCSL does not (yet) support algebraic specifications
- CoCasl does not mention bisimulations
- BObj — need to investigate

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