The Coalgebraic Class Specification Language CCSL

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Contents

I. Coalgebras — short intro
II. CCSL at work: A coalgebraic Queue specification
III. Modal logic
IV. Induction & Coinduction nested
V. Object-oriented Concepts
VI. Binary Methods
VII. Conclusion

Coalgebra is a function $X \longrightarrow \boxed{\cdots X \cdots}$

for instance $\mathsf{Aut}_{A,B} \xrightarrow{\langle out, next \rangle} A \Rightarrow (\mathsf{Aut}_{A,B} \times B)$

Morphisms of Coalgebras as you would expect:

$$\begin{array}{ccc} X & \xrightarrow{c} & F(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ Y & \xrightarrow{d} & F(Y) \end{array}$$

Important concepts

- Bisimulation $x \, R \, y \iff x$ and y show the same behaviour
  defined via morphisms [AM89] or via relation lifting [HJ98]
- Invariant if $x \in P$ then all successors of x are also in P
- final coalgebra most abstract and contains all possible behaviours

Coalgebras can model Classes of object-oriented languages

class Point
  methods
    get_x : **Self** $\longrightarrow$ real
    get_y : **Self** $\longrightarrow$ real
    move : **Self** $\times$ real $\times$ real $\longrightarrow$ **Self**
end Point

Coalgebra $c : X \longrightarrow T_P(X)$ with

$$T(X) = \mathbb{R} \times \mathbb{R} \times (\mathbb{R} \times \mathbb{R} \Rightarrow X)$$

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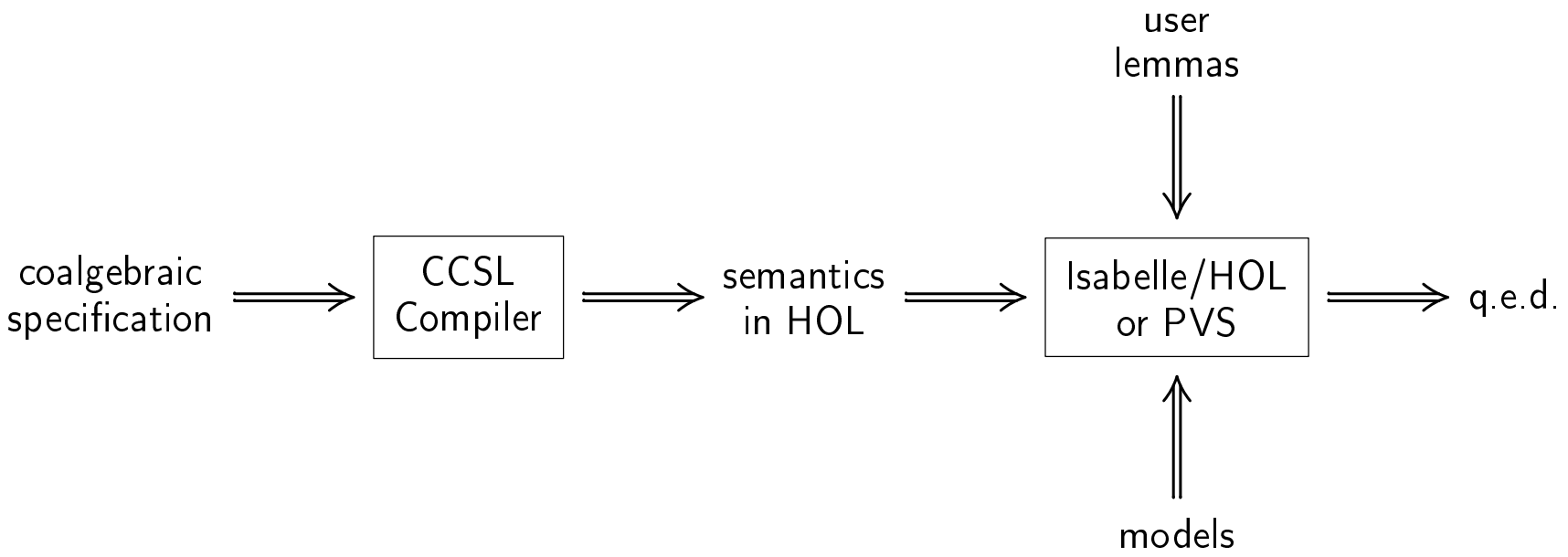
algebraic data types
$+$
coalgebraic class specifications
$+$
higher-order logic

$=$ CCSL

CCSL Compiler



Queue Example

```
Begin Queue[ A : Type ] : ClassSpec
  Method
    put : [Self, A] -> Self;
    top : Self -> Lift[[A,Self]];
  Constructor
    new : Self;
  Assertion Selfvar x : Self
    q_empty :
      x.top ~ bot Implies Forall(a : A) . x.put(a).top ~ up(a,x);
    q_filled :
      Forall(a1:A, y : Self) . x.top ~ up(a1,y) Implies
        Forall(a2 : A) . x.put(a2).top ~ up(a1, y.put(a2));
  Creation
    q_new : new.top ~ bot;
  Theorem Importing QueueModal[Self,A]
    strong_reachable : Forall(p, q : Self) :
      Let finite? : [Self -> bool] = Lambda(q : Self) :
              (Eventually Lambda(x : Self) : x.top = bot For {top}) q IN
        finite? p And finite? q  Implies
          (Eventually (Eventually Lambda(r : Self) : r ~ q For {put})
           For {top}
          ) p;
End Queue
```

```
Begin Queue[ A : Type ] : ClassSpec
  Method
    put : Self * A -> Self;
    top : Self -> Lift[A * Self];
  Constructor
    new : Self;
  Assertion Selfvar x : Self
    q_empty :
      x.top ~ bot Implies Forall(a : A) . x.put(a).top ~ up(a,x);
    q_filled :
      Forall(a1:A, y : Self) . x.top ~ up(a1,y) Implies
        Forall(a2 : A) . x.put(a2).top ~ up(a1, y.put(a2));
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    q_new : new.top ~ bot;
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              (Eventually Lambda(x : Self) : x.top = bot For {top}) q IN
        finite? p And finite? q  Implies
          (Eventually (Eventually Lambda(r : Self) : r ~ q For {put})
           For {top}
          ) p;
End Queue
```

- parametrised class specifications
  - inheritance (of specifications)
  - methods, constructor declarations
  - binary methods
  - attributes & update operations
  - final or loose semantics
  - axioms for methods and constructors
  - behavioural equality
  - method-wise infinitary modal operators
  - theorems
- parametrised abstract data types
  - initial semantics
  - no axioms
- ground signature extentions
  - parametric Types & Terms
  - definitions & declarations
- ADT's and Classes can be nested $\Longrightarrow$ nested induction and coinduction
- higher-order logic
- support for PVS and Isabelle/HOL
- coalgebraic refinement [JT01]
- Version 2.2 available at http://wwwtcs.inf.tu-dresden.de/~tews/ccsl

Invariant is a predicate closed under the coalgebra

in the Queue example: $q \in P \Longrightarrow \forall a \,.\, \mathsf{put}(q, a) \in P$

Invariants form a complete lattice

$\underline{\boldsymbol{P}}$ denotes the greatest invariant contained in P

$$\underline{P} \quad = \quad \bigvee \{Q \mid Q \subseteq P \ \wedge \ Q \text{ is an invariant}\}$$

Observe $x \in \underline{P}$ then P holds on all direct and indirect successor states of x

$\Box$ (always) acts on predicates (like a variable binder)

$$\Box : (\textsf{Self} \longrightarrow \textsf{Prop}) \longmapsto (\textsf{Self} \longrightarrow \textsf{Prop})$$

in the logic: $\Gamma, x : \textsf{Self} \vdash \varphi$ gives $\Gamma, y : \textsf{Self} \vdash (\Box \lambda x : \textbf{Self} \,.\, \varphi)\,(y)$.

Semantics

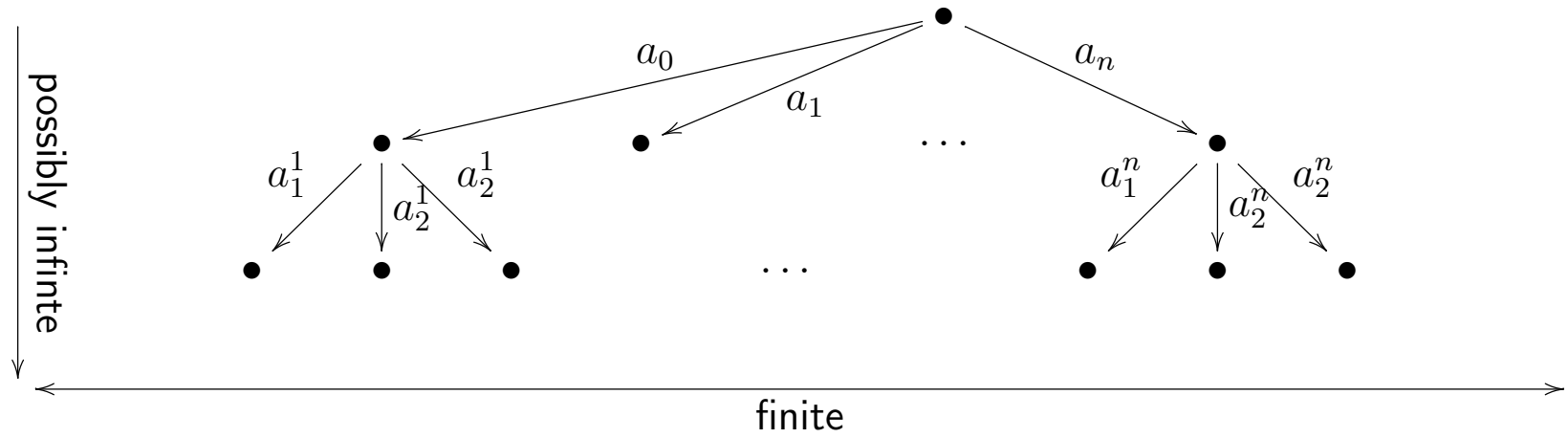
$$[\![\Box_x \varphi\,(y)]\!] \quad = \quad \left\{ (\vec{a}, b) \in [\![\Gamma]\!] \times [\![\textsf{Self}]\!] \;\mid\; b \in \underline{\{x \in [\![\textsf{Self}]\!] \;\mid\; [\![\varphi]\!](\vec{a}, x)\}} \right\}$$

Eventually $\Diamond\varphi \stackrel{\text{def}}{=} \neg\Box\neg\varphi \qquad \left(\stackrel{\text{def}}{=} \lambda x : \textbf{Self} \,.\, \neg(\Box \lambda y : \textbf{Self} \,.\, \neg\varphi(y))\,(x) \right)$

Method-wise modal operators arise from subcoalgebras

```
Begin list[ X : Type ] : Adt
   Constructor
      null : Carrier;
      cons( car, cdr ) : [X, Carrier] -> Carrier
End list

Begin InfTreeFin[A : Type] : Final ClassSpec
  Method
    next : Self -> list[[A,Self]];
End InfTreeFin
```



	Induction on lists	**Coinduction on InfTreeFin**
definition	Exposing a list algebra on a set X uniquely defines list$\longrightarrow X$	Exposing an InfTreeFin coalgebra structure on X uniquely defines $X\longrightarrow$InfTreeFin
proof	Any inductive predicate holds on all lists	Any InfTreeFin bisimulation is contained in the diagonal (i.e., $x\,R\,y \Longrightarrow x = y$)

Define empty_tree via $\mathsf{next}(\bot) = \mathsf{null}$

Define concat via $\mathsf{next}(p,q) = \begin{cases} \mathsf{map}(\lambda a, p'\,.\,(a,(p',q)))\,(\mathsf{next}\,p) & \text{if } \mathsf{next}\,p \neq \mathsf{null} \\ \mathsf{map}(\lambda a, q'\,.\,(a,(p,q')))\,(\mathsf{next}\,q) & \text{if } \mathsf{next}\,p = \mathsf{null} \end{cases}$

Prove $\mathsf{concat}(\mathsf{empty_tree}, p) \;=\; p$ (do induction inside a coinduction)

Validity investigated in [Hen99]

In general

- "object orientation" applies to software construction systems
- it does *not* apply to specification languages
- CCSL is a specification language *for* object-oriented programs
- CCSL gives general support —
  it does not specialise to the peculiar semantics of a special OO language

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CCSL supports

- multiple inheritance
- overriding of methods
- late binding possible
- object identity possible
- dynamic type checking
  (down casts) possible

Terminology

class specification

class is a model $\langle X, c : X \longrightarrow T(X) \rangle$

(state of an) object is an element $x \in X$

Inheritance of specifications

- inherits methods & method axioms
- renaming possible
- simple semantics: Let $\mathcal{B}$ inherit from $\mathcal{A}$, then

$$T_B(X) \quad = \quad T_A(X) \quad \times \quad \boxed{\text{new methods of } \mathcal{B}}$$

Signature model of $\mathcal{B}$: $\langle Y, d : Y \longrightarrow T_B(Y) \rangle$

Semantics of the method axioms wrt. d: $[\![\mathcal{M}_\mathcal{B}]\!]_d : Y \longrightarrow \text{Prop}$

$$[\![\mathcal{M}_\mathcal{B}]\!]_d \quad = \quad [\![\mathcal{M}_\mathcal{A}]\!]_{\pi_1 \circ d} \quad \wedge \quad \boxed{\text{new axioms of } \mathcal{B}}$$

- π_1 is like an upcast — but is does not convert the type
- For final semantics: $!_{\pi_1 \circ d}$ is an static upcast that converts the type of the objects
- Inheritance on specifications does not require inheritance of models

Program

```
class parent = object
  method m = 0
end
```

```
class heir = object
  inherit parent
  method m = 1
end
```

Simple approach

```
Begin SParent : ClassSpec
  Method
    m : Self -> nat;
  Assertion Selfvar x : Self
    p1 : x.m <= 1;
End SParent
```

```
Begin SHeir : Classspec
  Inherit From SParent
  Assertion Selfvar x : Self
    h1 : x.m = 1;
End SHeir
```

More complex approach

```
Begin SParent : ClassSpec
  Method
    dynamic_type : Self -> nat;
    m : Self -> nat;
  Assertion Selfvar x : Self
    p1 : x.m <= 1;
    p2 : dynamic_type(x) = 0
           Implies x.m = 0;

  Constructor
    new_parent : Self;
  Creation
    p3 : dynamic_type(new_parent) = 0
End SParent
```

```
Begin SHeir : ClassSpec
  Inherit From SParent
  Assertion Selfvar x : Self
    h1 : dynamic_type(x) = 1
           Implies x.m = 1;

  Constructor
    new_heir : Self;
  Creation
    h2 : dynamic_type(new_heir) = 1
End SHeir
```

Binary Methods—Example

Class A = {

method equal(x : A) = x == this ;

method memory(env : $\mathbb{N} \longrightarrow A$) = $\cdots$ env(5).equal(this) $\cdots$;

method strange(f : A $\longrightarrow$ A) = $\cdots$;

}

Assume, that X is the state space of the class

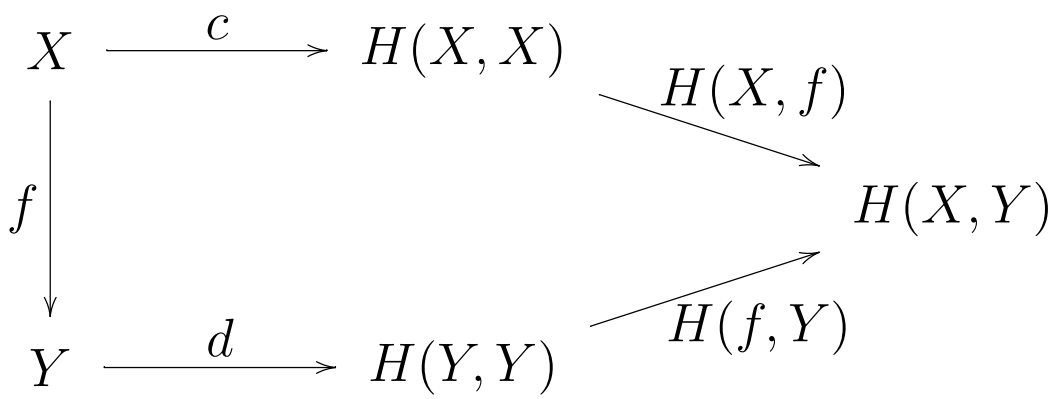
equal : $X \longrightarrow (X \longrightarrow \text{bool})$

memory : $X \longrightarrow (\mathbb{N} \longrightarrow X) \longrightarrow \boxed{\cdots}$

strange : $X \longrightarrow (X \longrightarrow X) \longrightarrow \boxed{\cdots}$

Endofunctors cannot describe such signatures!

- use bivariant functors $H : \mathbf{Set}^{op} \times \mathbf{Set} \longrightarrow \mathbf{Set}$
- Coalgebras are morphisms $X \xrightarrow{c} H(X, X)$
- Morphism of coalgebras:



- restrict to *polynomial functors*
  inductively built from constants, identity, $+$, $\times$, and $\Rightarrow$
- see [Tew01, Tew02b, Tew02a]

Extending polynomial functors

Class of Functors	Property	Example
cartesian $\mathbf{Set} \longrightarrow \mathbf{Set}$ $K \;=\; A \mid \mathsf{Id} \mid K_1 \overset{+}{\times} K_2$	no arguments	$\mathsf{Self} \longrightarrow \boxed{\cdots}$
polynomial $\mathbf{Set} \longrightarrow \mathbf{Set}$ $F \;=\; A \mid \mathsf{Id} \mid F_1 \overset{+}{\times} F_2 \mid A \Rightarrow F_1$	constant arguments	$\mathsf{Self} \times A \longrightarrow \boxed{\cdots}$
extended cartesian $\mathbf{Set}^{op} \times \mathbf{Set} \longrightarrow \mathbf{Set}$ $G^K \;=\; A \mid \mathsf{Id} \mid G_1^K \overset{+}{\times} G_2^K \mid K \Rightarrow G_1^K$	no functional arguments	$\mathsf{Self} \times \mathsf{Self} \times \mathsf{Self} \longrightarrow \boxed{\cdots}$ $\mathsf{Self} \times (\mathsf{Self} + A) \longrightarrow \boxed{\cdots}$
extended polynomial $\mathbf{Set}^{op} \times \mathbf{Set} \longrightarrow \mathbf{Set}$ $G^F \;=\; A \mid \mathsf{Id} \mid G_1^F \overset{+}{\times} G_2^F \mid F \Rightarrow G_1^F$	functional arguments with constant domain	$\mathsf{Self} \times (A \Rightarrow \mathsf{Self}) \longrightarrow \boxed{\cdots}$
higher-order polynomial $\mathbf{Set}^{op} \times \mathbf{Set} \longrightarrow \mathbf{Set}$ $H \;=\; A \mid \mathsf{Id} \mid H_1 \overset{+}{\times} H_2 \mid H_1 \Rightarrow H_2$	arbitrary arguments	$\mathsf{Self} \times (\mathsf{Self} \Rightarrow A) \longrightarrow \boxed{\cdots}$ $\mathsf{Self} \times (\mathsf{Self} \Rightarrow \mathsf{Self}) \longrightarrow \boxed{\cdots}$

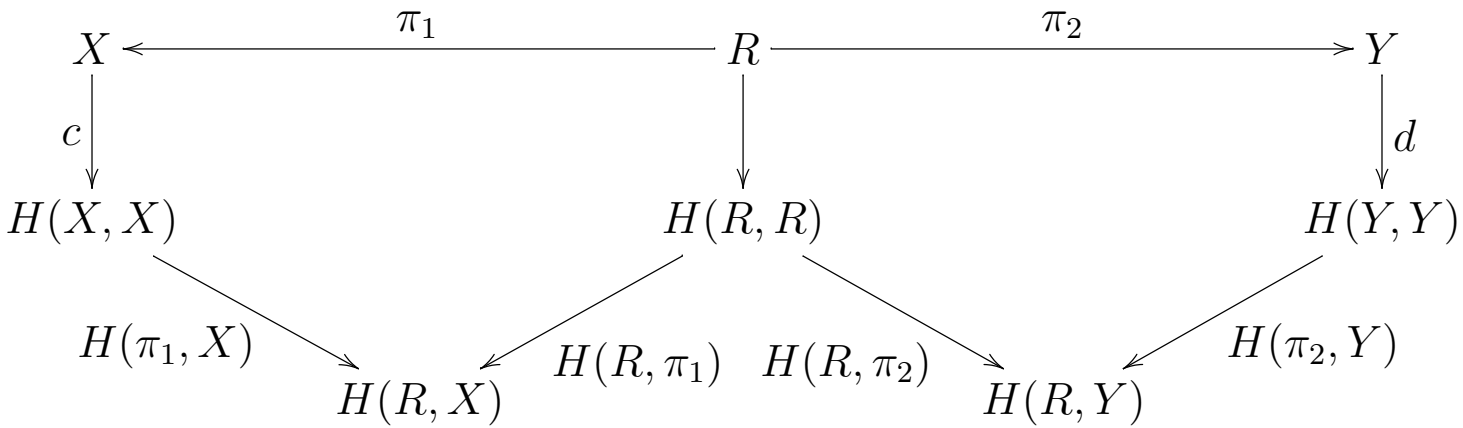
Consider a higher-order polynomial functor H and two coalgebras $c : X \longrightarrow H(X, X)$ and $d : Y \longrightarrow H(Y, Y)$.

Bisimulations according to Hermida & Jacobs [HJ98]

- R is a bisimulation for c and d, if

$$\forall x \in X, y \in Y \;.\; R(x, y) \quad \text{implies} \quad \mathrm{Rel}(H)(R, R)(c(x), d(y))$$

Bisimulations according to Aczel & Mendler [AM89]



Both notions coincide for extended polynomial functors
(in general they are different)

Relation lifting (for a h.-o. pol. functor H)

is an operation on relations

$$\big(S \subseteq U \times V, \quad R \subseteq X \times Y\big) \quad \longmapsto \quad \mathsf{Rel}(H)(S,R) \quad \subseteq \quad H(U,X) \times H(V,Y)$$

Defined by induction on the structure of H:

H	$\mathrm{Rel}(H)$	$a \in H(U,X) \quad b \in H(V,Y)$
X	R	$R(a,b)$
A	$=_A$	$a =_A b$
$H_1 \times H_2$	$\mathrm{Rel}(H_1) \times_{\mathrm{R}} \mathrm{Rel}(H_2)$	$\mathsf{Rel}(H_1)(S,R)(\pi_1\, a, \pi_1\, b) \ \wedge\ \mathsf{Rel}(H_2)(S,R)(\pi_2\, a, \pi_2\, b)$
$H_1 + H_2$	$\mathrm{Rel}(H_1) +_{\mathrm{R}} \mathrm{Rel}(H_2)$	$\begin{cases} \text{if } (a,b) = (\kappa_1\, a_1, \kappa_1\, b_1) & \mathsf{Rel}(H_1)(S,R)(a_1,b_1) \\ \text{if } (a,b) = (\kappa_2\, a_2, \kappa_2\, b_2) & \mathsf{Rel}(H_2)(S,R)(a_2,b_2) \\ \text{otherwise} & \bot \end{cases}$
$H_1 \Rightarrow H_2$	$\mathrm{Rel}(H_1) \Rightarrow_{\mathrm{R}} \mathrm{Rel}(H_2)$	$\forall (u,v) \in \mathsf{Rel}(H_1)(R,S) \ .\ \mathsf{Rel}(H_2)(S,R)(a(u), b(v))$

Extension

polynomial	structural properties + bisimulations form a *complete* lattice + greatest bisimulations + final models
extended cartesian	structural properties + bisimulations form an *incomplete* lattice + bisimulation equivalences form a *complete* lattice + greatest bisimulations equivalences for all coalgebras
extended polynomial	structural properties + bisimulations form an *incomplete* lattice + greatest bisimulations equivalences for finitely based coalgebras
higher-order	only trivial properties

- Results/Examples have been checked/developed with PVS.
  see `http://wwwtcs.inf.tu-dresden.de/~tews/binary/`
- can allow list : $\mathbf{Set} \longrightarrow \mathbf{Set}$

Let $G : \mathbf{Sets}^{op} \times \mathbf{Sets} \longrightarrow \mathbf{Sets}$ be a extended polynomial functor.

Proposition 1 *Relation lifting for G is fibred, i.e. commutes with substitution.*

Proposition 2 *Bisimulations and invariants are closed under arbitrary nonempty intersections.*

Proposition 3 *Bisimulations are closed under composition.*

Proposition 4 *The image $\coprod_f P$ of a morphism with respect to an invariant P is an invariant.*

Proposition 5 *Invariants correspond to subcoalgebras and vice versa.*

Proposition 6 *The predicate $\coprod_{\pi_1} R$ is an invariant for any bisimulation R.*

Proposition 7 *The relation $\pi_1^* P \cap R$ is a bisimulation for any invariant P and any bisimulation R.*

Proposition 8 *The predicate P is an invariant if and only if $\coprod_\delta P$ is a bisimulation.*

Proposition 9 *A function is a G–coalgebra morphism if and only if its graph is a bisimulation.*

Proposition 10 *The kernel of a G–morphism is a bisimulation.*

Proposition 11 *The Aczel/Mendler approach and the Hermida/Jacobs approach yield identical notions of invariant and bisimulation.*

Proposition 12 *If a class signature $\mathcal{A}$ contains a binary method m, then the associated signature functor $H_{\mathcal{A}}$ has a final coalgebra if and only if $H_{\mathcal{A}}(\mathbf{1},\mathbf{1})$ is a singleton.*

Proof

1. Assume H has a final coalgebra, say $z : Z \longrightarrow H(Z,Z)$.
2. Construct a new coalgebra $z' : Z+1 \longrightarrow H(Z+1, Z+1)$ such that
   - the inclusion $\iota : Z \longrightarrow Z+1$ is a coalgebra morphism *and*
   - for all $y \in Z$: $m_{z'}(\bot, \iota\, y) \neq m_z(y,y)$.
3. Note, that if $f : Z+1 \longrightarrow Z$ is a coalgebra morphism $z' \longrightarrow z$, then $f \circ \iota = \mathsf{id}_Z$.
4. Conclude, that there is no such morphism $z' \longrightarrow z$ by construction:

$$m_z(\circledast, \circledast) \quad = \quad m_z(\circledast, (f \circ \iota)\circledast) \quad = \quad m_{z'}(\bot, \iota\, \circledast)$$ □

Functor $K(Y,X) \stackrel{\text{def}}{=} Y \Rightarrow \text{bool}$

corresponding to one method equal : **Self** $\times$ **Self** $\longrightarrow$ bool

***K*–bisimulation** $R \subseteq X \times X \quad c : X \longrightarrow X \Rightarrow \text{bool}$

$$\forall x, x', y, y' : X \,.\, x\, R\, x' \;\wedge\; y\, R\, y' \;\Longrightarrow\; c\, x\, y \;=\; c\, x'\, y'$$

Defining a *K*–coalgebra $X \stackrel{\text{def}}{=} \{x_1, x_2\}$

$c \;:\; X \longrightarrow X \Rightarrow \text{bool} \qquad c\, x\, y \stackrel{\text{def}}{=} \text{if } x = y \text{ then } \top \text{ else } \bot \text{ endif}$

- $\mathrm{Eq}(X) = \{(x_1, x_1), (x_2, x_2)\}$ is a bisimulations for c
- $R \subseteq X \times X \stackrel{\text{def}}{=} \{(x_1, x_2), (x_2, x_1)\}$ is a bisimulation for c
- However, $\mathrm{Eq}(X) \cup R = X \times X$ is not a bisimulation

$$\top \;=\; c\, x_1\, x_1 \;\neq\; c\, x_1\, x_2 \;=\; \bot$$

Example: Greatest Bisimulation for EPF

Functor $G(Y,X) \stackrel{\text{def}}{=} (\mathbb{N} \Rightarrow Y) \Rightarrow \mathsf{bool}$

Defining a G–coalgebra on $\mathbb{N}$

$$c \;:\; \mathbb{N} \longrightarrow (\mathbb{N} \Rightarrow \mathbb{N}) \Rightarrow \mathsf{bool} \qquad c\,x\,f \;=\; \begin{cases} \top & \exists n\,.\,\forall i\,.\,f\,i < n \\ \bot & \text{otherwise} \end{cases}$$

G–bisimulation for c $R \subseteq \mathbb{N} \times \mathbb{N}$

$$\forall x, y : \mathbb{N},\, f, g : \mathbb{N} \longrightarrow \mathbb{N}\,.\, x\,R\,y \;\wedge\; \big(\forall n\,.\,(f\,n)\,R\,(g\,n)\big) \;\Longrightarrow\; c\,x\,f \;=\; c\,y\,g$$

Family $(S_n)_{n \in \mathbb{N}}$ $\qquad i\,S_n\,j \iff i = j \;\vee\; (i < n \wedge j < n)$

- all S_n are bisimulations
- every *finite* subset of S_n's has an upper bound (which is a bisimulation)
- $\bigcup_n S_n \;=\; \mathbb{N} \times \mathbb{N}$ is *not* a bisimulation, take $f_0 : \mathbb{N} \longrightarrow \mathbb{N}$ with $f_0\,n = 0$ and $\mathsf{id}_{\mathbb{N}}$: $\forall n\,.\,(n,0) \in \bigcup_n S_n$ but

$$\top \;=\; c\,0\,\mathsf{id}_{\mathbb{N}} \;\neq\; c\,0\,f_0 \;=\; \bot$$

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Applications

- MSMIE (multiprocessor shared-memory information exchange) protocol
- memory management in the

micro kernel operating system

Future work

- CCSL is not actively developed currently
- Application in the VFiasco project — Verification of Fiasco

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Visit `http://www.vfiasco.org`

- CCSL Version 3 (brainstorming)
  - general specifications with loose/initial/final semantics; multiple sorts
  - partial operations
  - generic specifications, signature morphisms

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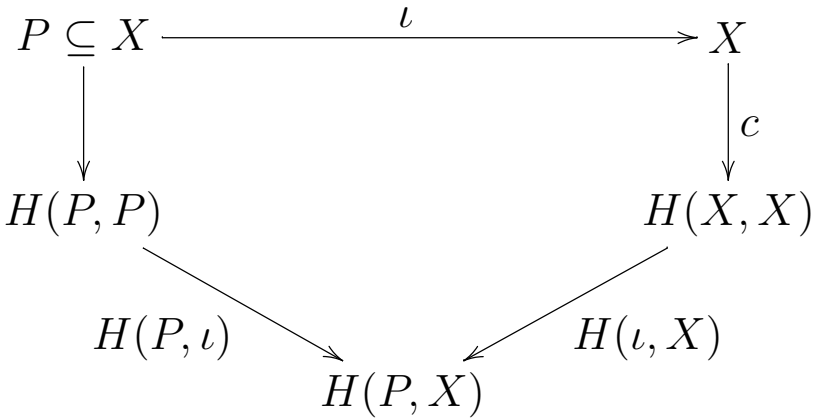
Consider a higher-order polynomial functor H and a coalgebras $c : X \longrightarrow H(X, X)$.

Invariants according to Hermida & Jacobs

- P is an invariant for c, if

$$\forall x \in X \,.\, P(x) \quad \text{implies} \quad \mathrm{Pred}(H)(P, P)(c(x))$$

Invariants according to Aczel & Mendler



Both notions coincide for extended polynomial functors
(in general they are different)

Predicate lifting is an operation on predicates

$$\big(Q \subseteq Y,\ \ P \subseteq X\big) \quad \longmapsto \quad \mathsf{Pred}(H)(Q,P) \quad \subseteq \quad H(Y,X)$$

Defined by induction on the structure of H:

H	$\mathsf{Pred}(H)(Q,P)(a)$ for $a \in H(Y,X)$
X	$P(a)$
A	$\top$
$H_1 \times H_2$	$\mathsf{Pred}(H_1)(Q,P)(\pi_1\, a) \ \wedge\ \mathsf{Pred}(H_2)(Q,P)(\pi_2\, a)$
$H_1 + H_2$	$\begin{cases} \text{if } a = \kappa_1\, a_1 & \mathsf{Pred}(H_1)(Q,P)(a_1) \\ \text{if } a = \kappa_2\, a_2 & \mathsf{Pred}(H_2)(Q,P)(a_2) \end{cases}$
$H_1 \Rightarrow H_2$	$\forall u \in \mathsf{Pred}(H_1)(P,Q)\ .\ \mathsf{Pred}(H_2)(Q,P)(a(u))$